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Abstract
In the classification of Hietarinta, three triangular 4×4 R-matrices lead, via the FRT formalism, to matrix bialgebras which are not deformations of the trivial one. In this paper, we find the bialgebras which are in duality with these three exotic matrix bialgebras. We note that the L–T duality of FRT is not sufficient for the construction of the bialgebras in duality. We find also the quantum planes corresponding to these bialgebras both by the Wess–Zumino R-matrix method and by Manin's method.

PACS numbers: 0210S, 0220, 1110, 1130

1. Introduction

Until recently it was not clear how many distinct quantum group deformations are admissible for the group $GL(2)$ and the supergroup $GL(1|1)$. For the group $GL(2)$ there were the well known standard $GL_{pq}(2)$ [1] and nonstandard (Jordanian) $GL_{gh}(2)$ [2] two-parameter deformations. For the supergroup $GL(1|1)$ there were the standard $GL_{pq}(1|1)$ [3–5] and the hybrid (standard–nonstandard) $GL_{qh}(1|1)$ [6] two-parameter deformations. (Various aspects of these matrix quantum (super-) group deformations were studied in, e.g., [7–38].) Recently, in [36] it was shown that the list of these four deformations is exhaustive (refuting a long-standing claim of [15] (supported also in [33, 38]) for the existence of a hybrid (standard–nonstandard) two-parameter deformation of $GL(2)$). In particular, it was shown that these four deformations match the distinct triangular 4×4 R-matrices from the classification of [39] which are deformations of the trivial R-matrix (corresponding to undeformed $GL(2)$).

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The matching mentioned above was performed by applying the FRT formalism [40] to these R-matrices. While applying this we noticed that a particular R-matrix, namely, the one denoted by $R_{H2,3}$ in [39], gives different results depending on the range of the three parameters h_1, h_2, h_3 it depends on. Only one of the ranges, namely, $h_1 = -h_2 = h$, $h_3 = -h^2$, contains the zero point which gives the trivial R-matrix. (It is a partial case of the two-parameter g, h Jordanian deformation for $g = -h$.) The other two ranges are given by $h_1 = -h_2 = h$, $h_3 \neq -h^2$ (*) and $h_1 \neq -h_2$ (**). Thus, the R-matrices obtained while varying through these ranges are not deformations of the trivial R-matrix, and also are distinct between each other. This analysis revealed altogether three distinctly different triangular R-matrices which are *not* deformations of the trivial R-matrix. In this way, in [36] three new matrix bialgebras which are *not* deformations of the classical algebra of functions over the group $GL(2)$ or the supergroup $GL(1|1)$ were obtained. These new matrix bialgebras, which we now call *exotic*, are very interesting and deserve further study. One of the first problems when dealing with such matrix bialgebras is to find the bialgebras with which they are in duality, since some of the structural characteristics are more transparent for the duals. The bialgebras in duality are also interesting objects with respect to the development of the representation theory.

This is the problem we solve in this paper. We find the bialgebras which are in duality with the three exotic matrix bialgebras found in [36]. We then find the quantum planes corresponding to these bialgebras by the Wess–Zumino R-matrix method [41] (cf also [42]). For the latter we find the minimal polynomials pol($\cdot$) in one variable such that pol($\hat{R}$) $= 0$ is the lowest-order polynomial identity satisfied by the singly permuted R-matrix $\hat{R} \equiv P R$ (P is the permutation matrix). These minimal polynomials indeed separate the three cases of $R_{H2,3}$ mentioned above. Namely, in case (*) we find a cubic minimal polynomial, while in case (**) it is quartic (cf (6.3), (6.4)). (Recall that the corresponding minimal polynomial in the Jordanian case is only quadratic.) We find also the quantum planes by Manin's method [7].

The paper is organized as follows. In section 2 we give a the overall general setting. Sections 3–5 are devoted to the separate study of the three exotic bialgebras. In each case we first give a more detailed (than in [36]) picture of the structure of these bialgebras, then we construct the bialgebra in duality, noting the bearing it has on the initial bialgebra; finally we show consistency of this approach to duality with the FRT one, noting the failings of the latter for these exotic bialgebras. In section 6 we construct the quantum planes corresponding to the three matrix bialgebras. Section 7 contains conclusions and outlook.

2. Exotic bialgebras: general setting

In this paper we consider the three exotic matrix bialgebras (obtained in [36]) which are *not* deformations of the classical algebra of functions over the group $GL(2)$ or the supergroup $GL(1|1)$. In all three cases these are unital associative algebras generated by four elements a, b, c, d which are not deformations of the classical algebra of functions over the group $GL(2)$ (or over the supergroup $GL(1|1)$). This is evident also from the algebraic relations which we give separately below. The coalgebraic relations are the classical ones:

$$\delta(T) = T \otimes T \qquad \varepsilon(T) = \mathbf{1}_2 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (2.1)$$

or explicitly

$$\delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \otimes a + b \otimes c & a \otimes b + b \otimes d \\ c \otimes a + d \otimes c & c \otimes b + d \otimes d \end{pmatrix} \qquad (2.2)$$

$$\varepsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (2.3)$$

However, the bialgebras under consideration are not Hopf algebras, as we shall show in each particular case.

3. Exotic bialgebras: case 1

3.1. Bialgebra relations

In this section we consider the matrix bialgebra, denoted here by $\mathcal{A}_1$, which is obtained by applying the *RTT* relations of [40]:

$$R\, T_1\, T_2 \;=\; T_2\, T_1\, R \tag{3.1}$$

where $T_1 \;=\; T \otimes \mathbf{1}_2$, $T_2 \;=\; \mathbf{1}_2 \otimes T$, for the case when $R \;=\; R_1$:

$$R_1 \;=\; \begin{pmatrix} 1 & h & -h & h_3 \\ & 1 & 0 & -h \\ & & 1 & h \\ & & & 1 \end{pmatrix} \qquad h_3 \neq -h^2. \tag{3.2}$$

This R-matrix together with the condition on the parameters is one of the special cases (mentioned in the introduction) of the R-matrix denoted by $R_{H2,3}$ in [39]. The algebraic relations of $\mathcal{A}_1$ obtained in this way are given by formulae (5.11) of [36], namely

$$
\begin{aligned}
c^2 &= 0 & ca &= ac = 0 & dc &= cd = 0 \\
da &= ad & cb &= bc & a^2 &= d^2 \\
ab &= ba + h(a^2 + bc - ad) & db &= bd - h(a^2 + bc - ad).
\end{aligned} \tag{3.3}
$$

Note that the constant h_3 does not enter the above relations.

Note that this bialgebra is not a Hopf algebra. Indeed, suppose that it is and there is an antipode γ, then we use one of the Hopf algebra axioms:

$$m \circ (\mathrm{id} \otimes \gamma) \circ \delta \;=\; i \circ \varepsilon \tag{3.4}$$

as maps $\mathcal{A} \to \mathcal{A}$, where m is the usual product in the algebra $m(Y \otimes Z) = YZ$, $Y, Z \in \mathcal{A}$, and i is the natural embedding of the number field F into $\mathcal{A}$: $i(c) = \mu 1_{\mathcal{A}}$, $\mu \in F$. Applying this to the element d we would have

$$c\,\gamma(b) + d\,\gamma(d) = 1_{\mathcal{A}}$$

which leads to contradiction after multiplying from the left by c (one would obtain $0 = c$).

The algebra $\mathcal{A}_1$ has the following PBW basis:

$$b^n a^k d^\ell \qquad b^n c \qquad n, k \in \mathbb{Z}_+ \quad \ell = 0, 1. \tag{3.5}$$

The last line of (3.3) strongly suggests the substitution

$$\tilde{a} \;=\; \tfrac{1}{2}(a + d) \qquad \tilde{d} \;=\; \tfrac{1}{2}(a - d) \tag{3.6}$$

so the new algebraic relations and PBW basis are

$$
\begin{aligned}
c^2 &= 0 & \tilde{a}c &= c\tilde{a} = \tilde{d}c = c\tilde{d} = \tilde{a}\tilde{d} = \tilde{d}\tilde{a} = 0 \\
cb &= bc & \tilde{a}b &= b\tilde{a} & \tilde{d}b &= b\tilde{d} + 2h\tilde{d}^2 + hbc
\end{aligned} \tag{3.7}
$$

$$b^n \tilde{a}^k \qquad b^n \tilde{d}^\ell \qquad b^n c \qquad n, k \in \mathbb{Z}_+ \quad \ell \in \mathbb{N}. \tag{3.8}$$

The coalgebra relations become

$$
\delta
\begin{pmatrix}
\tilde{a} \\
b \\
c \\
\tilde{d}
\end{pmatrix}
=
\begin{pmatrix}
\tilde{a} \otimes \tilde{a} + \tilde{d} \otimes \tilde{d} + \frac{1}{2} b \otimes c + \frac{1}{2} c \otimes b \\
\tilde{a} \otimes b + \tilde{d} \otimes b + b \otimes \tilde{a} - b \otimes \tilde{d} \\
c \otimes \tilde{a} + c \otimes \tilde{d} + \tilde{a} \otimes c - \tilde{d} \otimes c \\
\tilde{a} \otimes \tilde{d} + \tilde{d} \otimes \tilde{a} + \frac{1}{2} b \otimes c - \frac{1}{2} c \otimes b
\end{pmatrix}
\tag{3.9}
$$

$$
\varepsilon
\begin{pmatrix}
\tilde{a} & b \\
c & \tilde{d}
\end{pmatrix}
=
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}.
\tag{3.10}
$$

3.2. Duality

Two bialgebras $\mathcal{U}$, $\mathcal{A}$ are said to be *in duality* [43] if there exists a doubly nondegenerate bilinear form

$$
\langle \,,\, \rangle \ : \ \mathcal{U} \times \mathcal{A} \longrightarrow \mathbb{C} \qquad \langle \,,\, \rangle \ : \ (u, a) \mapsto \langle u, a \rangle \qquad u \in \mathcal{U} \quad a \in \mathcal{A}
\tag{3.11}
$$

such that, for $u, v \in \mathcal{U}, a, b \in \mathcal{A}$,

$$
\langle u, ab \rangle \ = \ \langle \delta_{\mathcal{U}}(u), a \otimes b \rangle \qquad \langle uv, a \rangle \ = \ \langle u \otimes v, \delta_{\mathcal{A}}(a) \rangle
\tag{3.12a}
$$

$$
\langle 1_{\mathcal{U}}, a \rangle \ = \ \varepsilon_{\mathcal{A}}(a) \qquad \langle u, 1_{\mathcal{A}} \rangle \ = \ \varepsilon_{\mathcal{U}}(u).
\tag{3.12b}
$$

Two Hopf algebras $\mathcal{U}$, $\mathcal{A}$ are said to be *in duality* [43] if they are in duality as bialgebras and if

$$
\langle \gamma_{\mathcal{U}}(u), a \rangle \ = \ \langle u, \gamma_{\mathcal{A}}(a) \rangle .
\tag{3.13}
$$

It is enough to define the pairing (3.11) between the generating elements of the two algebras. The pairing between any other elements of $\mathcal{U}$, $\mathcal{A}$ follows then from relations (3.12) and the standard bilinear form inherited by the tensor product.

The duality between two bialgebras or Hopf algebras may be used also to obtain the unknown dual of a known algebra. For that it is enough to give the pairing between the generating elements of the unknown algebra with arbitrary elements of the PBW basis of the known algebra. Using these initial pairings and the duality properties one may find the unknown algebra. One such possibility is given in [40]. However, their approach is not universal. In particular, it is not enough for the algebras considered here (as will become clear), and will be used only as a consistency check.

Another approach was initiated by Sudbery [12]. He obtained $U_q(sl(2)) \otimes U(u(1))$ as the algebra of tangent vectors at the identity of $GL_q(2)$. The initial pairings were defined through the tangent vectors at the identity. However, such calculations become very difficult for more complicated algebras. Thus, in [14] a generalization was proposed in which the initial pairings are postulated to be equal to the classical undeformed results. This generalized method was applied in [14] to the standard two-parameter deformation $GL_{p,q}(2)$ (where also Sudbery's method was used), then in [44] to the multiparameter deformation of $GL(n)$, in [45] to the matrix quantum Lorentz group of [46], in [27] to the Jordanian two-parameter deformation $GL_{g,h}(2)$, in [6] to the hybrid two-parameter deformation of the superalgebra $GL_{q,h}(1|1)$ and in [47] to the multiparameter deformation of the superalgebra $GL(m/n)$. (We note that the dual of $GL_{p,q}(2)$ was obtained also in [9] by methods of q-differential calculus.)

Let us denote by $\mathcal{U}_1$ the unknown yet dual algebra of $\mathcal{A}_1$, and by $\tilde{A}, B, C, \tilde{D}$ the four generators of $\mathcal{U}_1$. We would like as in [14] to define the pairing $\langle Z, f \rangle$, $Z = \tilde{A}, B, C, \tilde{D}$, f is from (3.8), as the classical tangent vector at the identity:

$$
\langle Z, f \rangle \ \equiv \ \varepsilon \left(\frac{\partial f}{\partial y} \right).
\tag{3.14}
$$

However, here this would work only for the pairs $(Z, y) = (\tilde{A}, \tilde{a}), (B, b), (\tilde{D}, \tilde{d})$ and not for (C, c). The reason is that classically some of the relations in (3.7) are constraints and we have to differentiate internally with respect to the manifold described by these constraints. In particular, if a constraint is given by setting $g = 0$, where g is some function of $\tilde{a}, b, c, \tilde{d}$, then any differentiation $\mathcal{D}$ should respect

$$(\mathcal{D} g f)_{g=0} = 0 \tag{3.15}$$

where f is any polynomial function of $\tilde{a}, b, c, \tilde{d}$. Thus, we are led to define

$$\langle C, f \rangle \equiv \varepsilon \left(E \frac{\partial}{\partial c} f \right) \tag{3.16}$$

where

$$E = \hat{E}\left(-\tilde{a}, \frac{\partial}{\partial \tilde{a}}\right) \tag{3.17a}$$

$$\hat{E}(x, y) \equiv \sum_{k=0}^{\infty} \frac{x^k y^k}{k!}. \tag{3.17b}$$

From the above definitions we obtain

$$\langle \tilde{A}, f \rangle = \varepsilon\left(\frac{\partial f}{\partial \tilde{a}}\right) = \delta_{n0} \begin{cases} k & \text{for} \quad f = b^n \tilde{a}^k \\ 0 & \text{for} \quad f = b^n \tilde{d}^\ell \\ 0 & \text{for} \quad f = b^n c \end{cases} \tag{3.18a}$$

$$\langle B, f \rangle = \varepsilon\left(\frac{\partial f}{\partial b}\right) = \delta_{n1} \begin{cases} 1 & \text{for} \quad f = b^n \tilde{a}^k \\ 0 & \text{for} \quad f = b^n \tilde{d}^\ell \\ 0 & \text{for} \quad f = b^n c \end{cases} \tag{3.18b}$$

$$\langle C, f \rangle = \varepsilon\left(E \frac{\partial f}{\partial c}\right) = \delta_{n0} \begin{cases} 0 & \text{for} \quad f = b^n \tilde{a}^k \\ 0 & \text{for} \quad f = b^n \tilde{d}^\ell \\ 1 & \text{for} \quad f = b^n c \end{cases} \tag{3.18c}$$

$$\langle \tilde{D}, f \rangle = \varepsilon\left(\frac{\partial f}{\partial \tilde{d}}\right) = \delta_{\ell 1}\delta_{n0} \begin{cases} 0 & \text{for} \quad f = b^n \tilde{a}^k \\ 1 & \text{for} \quad f = b^n \tilde{d}^\ell \\ 0 & \text{for} \quad f = b^n c \end{cases} \tag{3.18d}$$

$$\langle E, f \rangle = \begin{cases} 1 & \text{for} \quad f = 1_\mathcal{A} \\ 0 & \text{otherwise.} \end{cases} \tag{3.18e}$$

We have also included above the auxiliary generator E, since it will appear in the coproduct relations (cf below). Note that if we had taken the definition (3.14) for (C, c) the result in (3.18) would superficially be the same.

Now we can find the relations between the generators of $\mathcal{A}_1$. We have:

Proposition 1. *The generators $\tilde{A}, B, C, \tilde{D}, E$ introduced above obey the following relations:*

$$[\tilde{D}, C] = -2C \tag{3.19a}$$

$$[B, C] = \tilde{D} \tag{3.19b}$$

$$[B, C]_+ = \tilde{D}^2 \tag{3.19c}$$

$$[\tilde{D}, B] = 2B\tilde{D}^2 \tag{3.19d}$$

$$[\tilde{D}, B]_+ = 0 \tag{3.19e}$$

$$\tilde{D}^3 = \tilde{D} \tag{3.19f}$$

$$C^2 = 0 \qquad\qquad (3.19g)$$

$$[\tilde{A}, B] = 0 \qquad\qquad (3.19h)$$

$$[\tilde{A}, C] = 0 \qquad\qquad (3.19i)$$

$$[\tilde{A}, \tilde{D}] = 0 \qquad\qquad (3.19j)$$

$$EZ = ZE = 0 \qquad Z = \tilde{A}, B, C, \tilde{D}. \qquad\qquad (3.19k)$$

Proof. Using the assumed duality the above relations are shown by calculating their pairings with the basis monomials f of $\mathcal{A}_1$. In particular, we have

$$\langle C\tilde{D}, f \rangle = \begin{cases} 1 & \text{for} \quad f = c \\ 0 & \text{otherwise} \end{cases} \qquad\qquad (3.20a)$$

$$\langle \tilde{D}C, f \rangle = \begin{cases} -1 & \text{for} \quad f = c \\ 0 & \text{otherwise} \end{cases} \qquad\qquad (3.20b)$$

$$\langle BC, f \rangle = \begin{cases} \frac{1}{2} & \text{for} \quad f = \tilde{a} \\ \frac{1}{2} & \text{for} \quad f = \tilde{d} \\ 0 & \text{otherwise} \end{cases} \qquad\qquad (3.20c)$$

$$\langle CB, f \rangle = \begin{cases} \frac{1}{2} & \text{for} \quad f = \tilde{a} \\ -\frac{1}{2} & \text{for} \quad f = \tilde{d} \\ 0 & \text{otherwise} \end{cases} \qquad\qquad (3.20d)$$

$$\langle B\tilde{D}, f \rangle = \begin{cases} -1 & \text{for} \quad f = b \\ 0 & \text{otherwise} \end{cases} \qquad\qquad (3.20e)$$

$$\langle \tilde{D}B, f \rangle = \langle B\tilde{D}^2, f \rangle = \begin{cases} 1 & \text{for} \quad f = b \\ 0 & \text{otherwise} \end{cases} \qquad\qquad (3.20f)$$

$$\langle \tilde{D}^2, f \rangle = \begin{cases} 1 & \text{for} \quad f = a \\ 0 & \text{otherwise} \end{cases} \qquad\qquad (3.20g)$$

$$\langle \tilde{A}B, f \rangle = \langle B\tilde{A}, f \rangle = \begin{cases} k+1 & \text{for} \quad f = b\tilde{a}^k \\ 0 & \text{otherwise} \end{cases} \qquad\qquad (3.20h)$$

$$\langle \tilde{A}C, f \rangle = \langle C\tilde{A}, f \rangle = \begin{cases} 1 & \text{for} \quad f = c \\ 0 & \text{otherwise} \end{cases} \qquad\qquad (3.20i)$$

$$\langle \tilde{A}\tilde{D}, f \rangle = \langle \tilde{D}\tilde{A}, f \rangle = \begin{cases} 1 & \text{for} \quad f = \tilde{d} \\ 0 & \text{otherwise.} \end{cases} \qquad\qquad (3.20j)$$

The proposition now follows by formulae (3.20) and the defining relations (3.18). $\square$

We note that the algebraic relations (3.19) for $\mathcal{U}_1$ do not depend on the constant h present in the relations (3.7) of the dual algebra $\mathcal{A}_1$. Later, we shall see that the established duality reduces also the algebra $\mathcal{A}_1$ so that it also does not depend on h.

3.3. Coalgebra structure of the dual

We turn now to the coalgebra structure of $\mathcal{U}_1$. We have:

Proposition 2. *(i) The comultiplication in the algebra* $\mathcal{U}_1$ *is given by*

$$\delta(\tilde{A}) = \tilde{A} \otimes 1_{\mathcal{U}} + 1_{\mathcal{U}} \otimes \tilde{A} \tag{3.21a}$$

$$\delta(B) = B \otimes 1_{\mathcal{U}} + 1_{\mathcal{U}} \otimes B \tag{3.21b}$$

$$\delta(C) = C \otimes E + E \otimes C \tag{3.21c}$$

$$\delta(\tilde{D}) = \tilde{D} \otimes E + E \otimes \tilde{D} \tag{3.21d}$$

$$\delta(E) = E \otimes E. \tag{3.21e}$$

(ii) The co-unit relations in $\mathcal{U}_1$ *are given by*

$$\varepsilon_{\mathcal{U}}(Z) = 0 \qquad Z = \tilde{A}, B, C, \tilde{D} \tag{3.22a}$$

$$\varepsilon_{\mathcal{U}}(E) = 1 \tag{3.22b}$$

where we have included also the auxiliary operator E.

Proof. (i) We use the duality property (3.12a), namely we have

$$\langle Z, f_1 \, f_2 \rangle = \langle \delta_{\mathcal{U}}(Z), f_1 \otimes f_2 \rangle$$

for every generator Z of $\mathcal{U}_1$ and for every $f_1, f_2 \in \mathcal{A}_1$. Then we calculate separately the LHS and RHS and comparing the results prove (3.21).

(ii) Formulae (3.22) follow from $\varepsilon_{\mathcal{U}}(Z) = \langle Z, 1_{\mathcal{A}} \rangle$, cf (3.12b), and using the defining relations (3.18). □

There is no antipode for the bialgebra $\mathcal{U}_1$. Indeed, suppose that there was such. Then by applying the Hopf algebra axiom (3.4) to the generator E we would obtain

$$E \, \gamma(E) = 1_{\mathcal{U}}$$

which would lead to contradiction after multiplication from the left with $Z = \tilde{A}, B, C, \tilde{D}$ (we would obtain $0 = Z$).

3.4. Reduction of the bialgebra

We noticed that the algebraic relations (3.19) of $\mathcal{U}_1$ do not depend on the constant h from relations (3.7) of $\mathcal{A}_1$. The coproduct relations (3.21) do not depend on h either. We now clarify the reason for this. First we note that $\mathcal{A}_1$ has the following two-sided ideals and coideals:

$$I = \mathcal{A}_1 \, b\tilde{d} \oplus \mathcal{A}_1 \, \tilde{d}^2 \oplus \mathcal{A}_1 \, bc \tag{3.23a}$$

$$I_2 = \mathcal{A}_1 \, \tilde{d}^2 \oplus \mathcal{A}_1 \, bc \tag{3.23b}$$

$$I_1 = \mathcal{A}_1 \, bc \tag{3.23c}$$

so that

$$I_1 \subset I_2 \subset I \subset \mathcal{A}_1. \tag{3.24}$$

Furthermore the pairings of all these ideals with the dual algebra $\mathcal{U}_1$ vanish, thus we can set them consistently equal to zero. Thus, the basis of $\mathcal{A}_1$ is reduced to the following monomials:

$$b^n \tilde{a}^k \qquad n, k \in \mathbb{Z}_+ \qquad \tilde{d} \qquad c. \tag{3.25}$$

Actually, it was only these monomials that appeared in the proof of the dual relations (3.19). The algebraic relations of the reduced algebra become rather trivial:

$$\tilde{a}c = c\tilde{a} = \tilde{d}c = c\tilde{d} = \tilde{a}\tilde{d} = \tilde{d}\tilde{a} = cb = bc = \tilde{d}b = b\tilde{d} = 0$$
$$c^2 = 0 \qquad \tilde{a}b = b\tilde{a} \tag{3.26}$$

while the coalgebra relations remain unchanged and nontrivial. It is remarkable that the dual algebra has much richer structure in both the algebraic and coalgebraic sectors.

3.5. Consistency with the FRT approach

For the application of the FRT approach to duality we need the 4×4 R-matrix, which for the algebra $\mathcal{A}_1$ is given by (3.2). In the duality relations the matrices $R_1^{\pm}$ actually enter:

$$R_1^+ \equiv P\,R_1\,P = R_1(-h) = \begin{pmatrix} 1 & -h & h & h_3 \\ & 1 & 0 & h \\ & & 1 & -h \\ & & & 1 \end{pmatrix} \tag{3.27a}$$

$$R_1^- \equiv R_1^{-1} = \begin{pmatrix} 1 & -h & h & -h_3 - 2h^2 \\ & 1 & 0 & h \\ & & 1 & -h \\ & & & 1 \end{pmatrix} \tag{3.27b}$$

where P is the permutation matrix:

$$P \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.28}$$

These R-matrices encode (part of) the duality between $\mathcal{U}_1$ and $\mathcal{A}_1$ by formula (2.1) of [40] taken for $k = 1$ and written in our setting:

$$\langle L^{\pm}, T \rangle = R_1^{\pm} \tag{3.29}$$

where $L^{\pm}$ are 2×2 matrices whose elements are functions of the generators of $\mathcal{U}_1$ and T is the 2×2 matrix formed by the generators of $\mathcal{A}_1$, cf (2.1). In order to make formula (3.29) explicit we have to adopt some convention on the indices. We choose to write it as

$$\langle L_{ik}^{\pm}, T_{\ell j} \rangle = (R_1^{\pm})_{ijk\ell} \qquad i, j, k, \ell = 1, 2 \tag{3.30}$$

where the enumeration of the R-matrices is done as in [8], namely the rows are enumerated from top to bottom by the pairs $(i, j) = (1, 1), (1, 2), (2, 1), (2, 2)$, and the columns are enumerated from left to right by the pairs $(k, \ell) = (1, 1), (1, 2), (2, 1), (2, 2)$.

Using all this and rewriting the result in terms of the new basis (3.7) of $\mathcal{A}_1$ we have

$$\left\langle L_{11}^{\pm}, \begin{pmatrix} \tilde{a} & b \\ c & \tilde{d} \end{pmatrix} \right\rangle = \left\langle L_{22}^{\pm}, \begin{pmatrix} \tilde{a} & b \\ c & \tilde{d} \end{pmatrix} \right\rangle = \begin{pmatrix} 1 & -h \\ 0 & 0 \end{pmatrix} \tag{3.31}$$

$$\left\langle L_{12}^{\pm}, \begin{pmatrix} \tilde{a} & b \\ c & \tilde{d} \end{pmatrix} \right\rangle = \begin{pmatrix} h & h_{\pm} \\ 0 & 0 \end{pmatrix} \tag{3.32}$$

where $h_+ = h_3$ and $h_- = -h_3 - 2h^2$. Note that the elements $L_{21}^{\pm}$ have zero products with all generators so we can set them to zero. Next we calculate the pairings with arbitrary elements of $\mathcal{A}_1$ for which we use the fact that the coproducts of the $L_{jk}^{\pm}$ generators are canonically given by [40]

$$\delta\left(L_{ik}^{\pm}\right) = \sum_{j=1}^{2} L_{ij}^{\pm} \otimes L_{jk}^{\pm}. \tag{3.33}$$

Using this we obtain

$$\langle L_{11}^{\pm}, b^n \tilde{a}^k \rangle = \langle L_{22}^{\pm}, b^n \tilde{a}^k \rangle = (-h)^n \tag{3.34}$$

$$\langle L_{12}^{\pm}, b^n \tilde{a}^k \rangle = (-1)^n h^{n-1}((k+n)h^2 - n(h_{\pm} + h^2)). \tag{3.35}$$

All other pairings are zero.

Computing the above pairings with the defining relations (3.18) we conclude that these L operators are expressed in terms of the generators of the dual algebra $\mathcal{U}_1$ as follows:

$$L_{11}^{\pm} = L_{22}^{\pm} = \mathrm{e}^{-hB} \tag{3.36a}$$

$$L_{12}^{\pm} = ((h_{\pm} + h^2)B + h\tilde{A})\mathrm{e}^{-hB} \tag{3.36b}$$

where expressions such as e^{vB} are defined as formal power series $\mathrm{e}^{vB} = 1_{\mathcal{U}} + \sum_{p \in \mathbb{Z}_+} \frac{v^p}{p!} B^p$. Formulae (3.36) are compatible with the coproducts (3.21a) and (3.21b) of the generators $\tilde{A}$, B. However, as we see this approach does not say anything about the generators C, $\tilde{D}$.

4. Exotic bialgebras: case 2

4.1. Bialgebra relations

In this section we consider the bialgebra, denoted here by $\mathcal{A}_2$, which is obtained by applying the basic relations (3.1) for the case when $R = R_2$:

$$R_2 = \begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ & 1 & 0 & h_2 \\ & & 1 & h_1 \\ & & & 1 \end{pmatrix} \qquad h_1 + h_2 \neq 0. \tag{4.1}$$

This R-matrix together with the condition on the parameters is the second of the special cases (mentioned in the introduction) of the R-matrix denoted by $R_{H2,3}$ in [39]. Its algebraic relations thus obtained are given by formulae (5.9) of [36], namely

$$\begin{aligned}
c^2 &= 0 & ca &= ac = 0 & dc &= cd = 0 \\
da &= ad & cb &= bc \\
a^2 &= d^2 = ad + bc \\
ab &= bd = ba + (h_1 - h_2)bc & db &= bd + (h_2 - h_1)bc.
\end{aligned} \tag{4.2}$$

Note that the constant h_3 does not enter the above relations.

The coalgebra relations are the same as for $\mathcal{A}_1$. Also the demonstration that this bialgebra is not a Hopf algebra is performed as for $\mathcal{A}_1$. The PBW basis in this case is

$$b^n a^k \qquad a^\ell d \qquad c \qquad n, k \in \mathbb{Z}_+ \quad \ell = 0, 1. \tag{4.3}$$

Also in this case we make the change of basis (3.6) to obtain

$$\begin{aligned}
c^2 &= 0 & \tilde{a}c &= c\tilde{a} = \tilde{d}c = c\tilde{d} = \tilde{a}\tilde{d} = \tilde{d}\tilde{a} = 0 \\
\tilde{a}b &= b\tilde{a} \\
bc &= cb = 2\tilde{d}^2 & \tilde{d}^3 &= 0 \\
\tilde{d}b &= -b\tilde{d} = (h_1 - h_2)\tilde{d}^2.
\end{aligned} \tag{4.4}$$

The PBW basis becomes

$$b^n \tilde{a}^k \qquad \tilde{d}^\ell \qquad c \qquad n, k \in \mathbb{Z}_+ \quad \ell = 1, 2. \tag{4.5}$$

Thus, this bialgebra looks 'smaller' than $\mathcal{A}_1$—compare with (4.5). It has also a smaller structure of two-sided ideals and coideals:

$$I_2 = \mathcal{A}_2 \tilde{d}^2 \oplus \mathcal{A}_2 bc \tag{4.6a}$$

$$I_1 = \mathcal{A}_2 bc \tag{4.6b}$$

so that

$$I_1 \subset I_2 \subset \mathcal{A}_2 \tag{4.7}$$

(compare with (3.23) and (3.24)).

4.2. Algebra and coalgebra structure of the dual

In view of the similarities between the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ it is natural to use the same generators $\tilde{A}, B, C, \tilde{D}, E$ for the dual $\mathcal{U}_2$. It is not surprising that we obtain the same algebraic and coalgebraic relations. We have:

Proposition 3. *The generators* $\tilde{A}, B, C, \tilde{D}, E$ *of the bialgebra* $\mathcal{U}_2$ *obey the same algebraic and coalgebraic relations as for the algebra* $\mathcal{U}_1$ *given in propositions 1 and 2.*

Proof. The proof is based on the fact that the bialgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ differ in the relations involving the (co-) ideals I_k, which have no bearing on the relations of $\mathcal{U}_1$. Thus, we need only to show that all bilinears built from the generators $\tilde{A}, B, C, \tilde{D}, E$ have zero pairings with the ideals I_k (cf (4.6), (4.7)), which is easy to demonstrate. $\square$

As a corollary also here the basis and algebraic relations of $\mathcal{A}_2$ reduce to (3.25) and (3.26).

Thus, we have the following important conclusion:

Proposition 4. *The bialgebras* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *considered as bialgebras in duality with the bialgebras* $\mathcal{U}_1$ *and* $\mathcal{U}_2$, *respectively, coincide.*

We recall that the notion of duality we use does not coincide with the FRT definition of duality. The latter is more stringent as we shall see in the next subsection.

4.3. Consistency with the FRT approach

The 4×4 R-matrix needed for the FRT approach is given in (4.1). The matrices $R_2^{\pm}$ entering the duality relations are

$$R_2^+ \equiv P R_2 P = \begin{pmatrix} 1 & h_2 & h_1 & h_3 \\ & 1 & 0 & h_1 \\ & & 1 & h_2 \\ & & & 1 \end{pmatrix} \tag{4.8a}$$

$$R_2^- \equiv R_2^{-1} = \begin{pmatrix} 1 & -h_1 & -h_2 & 2h_1 h_2 - h_3 \\ & 1 & 0 & -h_2 \\ & & 1 & -h_1 \\ & & & 1 \end{pmatrix}. \tag{4.8b}$$

Using the above and relations (3.30) (with $R_1 \to R_2$) we obtain

$$\left\langle L_{11}^+, \begin{pmatrix} \tilde{a} & b \\ c & \tilde{d} \end{pmatrix} \right\rangle = \left\langle L_{22}^+, \begin{pmatrix} \tilde{a} & b \\ c & \tilde{d} \end{pmatrix} \right\rangle = \begin{pmatrix} 1 & h_2 \\ 0 & 0 \end{pmatrix} \tag{4.9}$$

$$\left\langle L_{12}^+, \begin{pmatrix} \tilde{a} & b \\ c & \tilde{d} \end{pmatrix} \right\rangle = \begin{pmatrix} h_1 & h_3 \\ 0 & 0 \end{pmatrix} \tag{4.10}$$

$$\left\langle L_{11}^-, \begin{pmatrix} \tilde{a} & b \\ c & \tilde{d} \end{pmatrix} \right\rangle = \left\langle L_{22}^-, \begin{pmatrix} \tilde{a} & b \\ c & \tilde{d} \end{pmatrix} \right\rangle = \begin{pmatrix} 1 & -h_1 \\ 0 & 0 \end{pmatrix} \tag{4.11}$$

$$\left\langle L_{12}^-, \begin{pmatrix} \tilde{a} & b \\ c & \tilde{d} \end{pmatrix} \right\rangle = \begin{pmatrix} -h_2 & -h_3 + 2h_1 h_2 \\ 0 & 0 \end{pmatrix}. \tag{4.12}$$

Iterating this we obtain

$$\left\langle L_{11}^+, b^n \tilde{a}^k \right\rangle = \left\langle L_{22}^+, b^n \tilde{a}^k \right\rangle = h_2^n \tag{4.13}$$

$$\left\langle L_{12}^+, b^n \tilde{a}^k \right\rangle = h_2^{n-1}((k+n)h_1 h_2 + n(h_3 - h_1 h_2)) \tag{4.14}$$

$$\left\langle L_{11}^-, b^n \tilde{a}^k \right\rangle = \left\langle L_{22}^-, b^n \tilde{a}^k \right\rangle = (-h_1)^n \tag{4.15}$$

$$\left\langle L_{12}^-, b^n \tilde{a}^k \right\rangle = (-h_1)^{n-1}((k+n)h_1 h_2 + n(-h_3 + h_1 h_2)). \tag{4.16}$$

From the above follow

$$L_{11}^+ = L_{22}^+ = e^{h_2 B} \tag{4.17a}$$

$$L_{12}^+ = ((h_3 - h_1 h_2)B + h_1 \tilde{A})e^{h_2 B} \tag{4.17b}$$

$$L_{11}^- = L_{22}^+ = e^{-h_1 B} \tag{4.17c}$$

$$L_{12}^- = ((-h_3 + h_1 h_2)B - h_2 \tilde{A})e^{-h_1 B}. \tag{4.17d}$$

This is compatible with the coproducts for the operators $\tilde{A}$, B.

Thus, we see that the L operators in this case are different from those of $\mathcal{U}_1$ (cf (3.36)). Thus, the FRT approach is more stringent than the notion of duality we use since it distinguishes the two pairs of bialgebras. However, this difference is not as drastic as the difference between the algebraic relations (3.7) and (4.4) of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, since (3.36) is just a special case of (4.17) obtained for $h_1 = -h_2 = h$. On the other hand the FRT approach is incomplete in the cases at hand since it gives information only about some of the generators, namely, $\tilde{A}$ and B, and says nothing about the generators C, $\tilde{D}$.

5. Exotic bialgebras: case 3

5.1. Bialgebra relations

In this section we consider the bialgebra which we denote here by $\mathcal{A}_3$. It is obtained by applying the basic relations (3.1) for the case when $R = R_3$:

$$R_3 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ & -1 & 0 & 0 \\ & & -1 & 0 \\ & & & 1 \end{pmatrix}. \tag{5.1}$$

This R-matrix is denoted by $R_{S0,2}$ in [39]. The algebraic relations of $\mathcal{A}_3$ are given by formulae (5.13) of [36], namely

$$
\begin{array}{lll}
c^2 = 0 & ca = ac = 0 & dc = cd = 0 \\
da = ad & cb = bc & a^2 = d^2 \\
ab + ba = 0 & db + bd = 0.
\end{array} \tag{5.2}
$$

The coalgebra relations and the demonstration that this bialgebra is not a Hopf algebra are as for $\mathcal{A}_1$, $\mathcal{A}_3$.

Also in this case we make the change of basis (3.6) to obtain

$$
\begin{array}{lll}
c^2 = 0 & \tilde{a}c = c\tilde{a} = \tilde{d}c = c\tilde{d} = \tilde{a}\tilde{d} = \tilde{d}\tilde{a} = 0 & cb = bc \\
\tilde{a}b + b\tilde{a} = 0 & \tilde{d}b + b\tilde{d} = 0.
\end{array} \tag{5.3}
$$

The algebra $\mathcal{A}_3$ has the same PBW bases (3.5) and (3.8) as the algebra $\mathcal{A}_1$. It has also the same (co-) ideals as $\mathcal{A}_1$ (cf (3.23), (3.24)).

5.2. Algebra and coalgebra structure of the dual

In view of the similarities between the algebras $\mathcal{A}_1$ and $\mathcal{A}_3$ it is natural to use the same generators $\tilde{A}, B, C, \tilde{D}, E$ for the dual $\mathcal{U}_3$. It is not surprising that we obtain the same algebraic relations between generators $\tilde{A}, B, C, \tilde{D}, E$. However, unlike the bialgebras $\mathcal{A}_1$, $\mathcal{A}_2$, the coalgebraic relations and the relation with the FRT formalism here are different and it is even necessary to introduce two new auxilliary operators $F_\pm$ defined as

$$\langle F_\pm, f \rangle \equiv \varepsilon \left(\hat{E} \left(\pm 1, \frac{\partial}{\partial \tilde{d}} \right) f \right) = \varepsilon \left(\exp \left(\pm \frac{\partial}{\partial \tilde{d}} \right) f \right). \tag{5.4}$$

Explicitly we have

$$\langle F_+, f \rangle = \begin{cases} 1 & \text{for} \quad f = \tilde{d}^\ell \\ 1 & \text{for} \quad f = 1_A \\ 0 & \text{otherwise} \end{cases} \tag{5.5a}$$

$$\langle F_-, f \rangle = \begin{cases} (-1)^\ell & \text{for} \quad f = \tilde{d}^\ell \\ 1 & \text{for} \quad f = 1_A \\ 0 & \text{otherwise.} \end{cases} \tag{5.5b}$$

We have for the algebraic and coalgebraic structure of $\mathcal{U}_3$:

Proposition 5. *The generators $\tilde{A}, B, C, \tilde{D}, E, F_\pm$ obey the following algebraic relations:*

$$[\tilde{D}, C] = -2C \tag{5.6a}$$

$$[B, C] = \tilde{D} \tag{5.6b}$$

$$[B, C]_+ = \tilde{D}^2 \tag{5.6c}$$

$$[\tilde{D}, B] = 2B\tilde{D}^2 \tag{5.6d}$$

$$[\tilde{D}, B]_+ = 0 \tag{5.6e}$$

$$\tilde{D}^3 = \tilde{D} \tag{5.6f}$$

$$C^2 = 0 \tag{5.6g}$$

$$[\tilde{A}, B] = 0 \tag{5.6h}$$

$$[\tilde{A}, C] = 0 \tag{5.6i}$$

$$[\tilde{A}, \tilde{D}] = 0 \tag{5.6j}$$

$$EZ = ZE = 0 \qquad Z = \tilde{A}, B, C, \tilde{D} \tag{5.6k}$$

$$F_+^2 = F_-^2 = 1_{\mathcal{U}} \tag{5.6l}$$

$$[F_+, F_-] = 0 \tag{5.6m}$$

$$[\tilde{A}, F_\pm] = 0 \tag{5.6n}$$

$$BF_\pm \pm F_\mp B = 0 \tag{5.6o}$$

$$[C, F_\pm]_+ = 0 \tag{5.6p}$$

$$[\tilde{D}, F_\pm] = 0 \tag{5.6q}$$

$$EF_\pm = F_\pm E = E. \tag{5.6r}$$

Proof. The relations between the generators $\tilde{A}, B, C, \tilde{D}, E$ are the same as for the algebra $\mathcal{U}_1$ and they are proved in the same way using (3.20). The only difference may exist in (3.20h), since b and $\tilde{a}$ anticommute, but what is essential is that $\delta(b\tilde{a}^k)$ is symmetric when paired with $\tilde{A}B$ and $B\tilde{A}$ (the asymmetric terms involve bc and $\tilde{d}^2$ and give no contribution). Verifying the relations involving $F_\pm$ is a straightforward calculation. $\square$

Proposition 6. *(i) The comultiplication in the algebra $\mathcal{U}_3$ is given by*

$$\delta(\tilde{A}) = \tilde{A} \otimes 1_{\mathcal{U}} + 1_{\mathcal{U}} \otimes \tilde{A} \tag{5.7a}$$

$$\delta(B) = B \otimes 1_{\mathcal{U}} + F_+ F_- \otimes B \tag{5.7b}$$

$$\delta(C) = C \otimes E + E \otimes C \tag{5.7c}$$

$$\delta(\tilde{D}) = \tilde{D} \otimes E + E \otimes \tilde{D} \tag{5.7d}$$

$$\delta(E) = E \otimes E \tag{5.7e}$$

$$\delta(F_\pm) = F_\pm \otimes F_\pm. \tag{5.7f}$$

(ii) The co-unit relations in $\mathcal{U}_3$ are given by

$$\varepsilon_{\mathcal{U}}(Z) = 0 \qquad Z = \tilde{A}, B, C, \tilde{D} \qquad (5.8a)$$

$$\varepsilon_{\mathcal{U}}(Z) = 1 \qquad Z = E, F_{\pm}. \qquad (5.8b)$$

Proof. The proof is by the same methods as that of proposition 2. $\qquad\square$

There is no antipode for the bialgebra $\mathcal{U}_3$—this is proved exactly as for $\mathcal{U}_1$.

As in the case of $\mathcal{U}_1 \leftrightarrow \mathcal{A}_1$ (and $\mathcal{U}_2 \leftrightarrow \mathcal{A}_2$) duality one may reduce the basis of $\mathcal{A}_3$ from the $\mathcal{U}_3 \leftrightarrow \mathcal{A}_3$ duality, but only with the ideal $I_1 = \mathcal{A}_3 \, bc$ (since $\tilde{d}^2$ is not annihilated by $F_{\pm}$). Thus, the basis of $\mathcal{A}_3$ is reduced to the following monomials:

$$b^n \tilde{a}^k \qquad b^n \tilde{d}^{\ell} \qquad c \qquad n, k \in \mathbb{Z}_+ \quad \ell \in \mathbb{N}. \qquad (5.9)$$

The algebraic relations of the reduced algebra become

$$c^2 = 0 \qquad \tilde{a}c = c\tilde{a} = \tilde{d}c = c\tilde{d} = \tilde{a}\tilde{d} = \tilde{d}\tilde{a} = cb = bc = 0$$
$$\tilde{a}b + b\tilde{a} = 0 \qquad \tilde{d}b + b\tilde{d} = 0. \qquad (5.10)$$

5.3. Consistency with the FRT approach

The 4×4 R-matrix needed for the FRT approach is given in (5.1). The matrices $R_3^{\pm}$ entering the duality relations are:

$$R_3^+ \equiv P \, R_3 \, P = R_3 \qquad (5.11a)$$

$$R_3^- \equiv R_3^{-1} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ & -1 & 0 & 0 \\ & & -1 & 0 \\ & & & 1 \end{pmatrix}. \qquad (5.11b)$$

Using the above and relations (3.30) (with $R_1 \to R_3$) we obtain

$$\left\langle L_{11}^{\pm}, \begin{pmatrix} \tilde{a} & b \\ c & \tilde{d} \end{pmatrix} \right\rangle = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad (5.12)$$

$$\left\langle L_{22}^{\pm}, \begin{pmatrix} \tilde{a} & b \\ c & \tilde{d} \end{pmatrix} \right\rangle = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \qquad (5.13)$$

$$\left\langle L_{12}^{\pm}, \begin{pmatrix} \tilde{a} & b \\ c & \tilde{d} \end{pmatrix} \right\rangle = \begin{pmatrix} 0 & \pm 1 \\ 0 & 0 \end{pmatrix}. \qquad (5.14)$$

Iterating these relations for arbitrary elements of the basis of $\mathcal{A}_3$ we can show that the L generators are given in terms of some of the other generators in the following way:

$$L_{11}^{\pm} = F_+ \qquad L_{22}^{\pm} = F_- \qquad L_{12}^{\pm} = \pm B F_-. \qquad (5.15)$$

Formulae (5.15) are compatible with the coproducts in (5.7) of the generators $B, F_{\pm}$. However, as we see this approach does not say anything about the basic generators $\tilde{A}, C, \tilde{D}$.

6. Higher-order R-matrix relations and quantum planes

In order to address the question of the quantum planes corresponding to the exotic bialgebras we have to know the relations which the R-matrices fulfil. As we know the R-matrices producing deformations of the $GL(2)$ and $GL(1|1)$ fulfil second-order relations. However, in the cases at hand we have higher-order relations.

We start with the R-matrix $R_{H2,3}$ of [39]:

$$R = \begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ & 1 & 0 & h_2 \\ & & 1 & h_1 \\ & & & 1 \end{pmatrix}. \tag{6.1}$$

We need actually the singly permuted R-matrix:

$$\hat{R} \equiv P R = \begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ 0 & 0 & 1 & h_1 \\ 0 & 1 & 0 & h_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.2}$$

Explicit calculation shows now that we have

$$(\hat{R} - \mathbf{1})\,(\hat{R} + \mathbf{1}) \;=\; 0 \qquad h_1 = -h_2 = h \qquad h_3 = -h^2 \tag{6.3a}$$

$$(\hat{R} - \mathbf{1})^2\,(\hat{R} + \mathbf{1}) \;=\; 0 \qquad h_1 = -h_2 = h \qquad h_3 \neq -h^2 \qquad \hat{R} = P R_1 \tag{6.3b}$$

$$(\hat{R} - \mathbf{1})^3\,(\hat{R} + \mathbf{1}) \;=\; 0 \qquad h_1 + h_2 \neq 0 \qquad \hat{R} = P R_2 \tag{6.3c}$$

where $\mathbf{1}$ is the 4×4 unit matrix. Thus the minimal polynomials are

$$\mathrm{pol}(\hat{R}) = \begin{cases} (\hat{R} - \mathbf{1})\,(\hat{R} + \mathbf{1}) & \text{for} \quad h_1 = -h_2 = h \qquad h_3 = -h^2 \\ (\hat{R} - \mathbf{1})^2\,(\hat{R} + \mathbf{1}) & \text{for} \quad h_1 = -h_2 = h \qquad h_3 \neq -h^2 \\ (\hat{R} - \mathbf{1})^3\,(\hat{R} + \mathbf{1}) & \text{for} \quad h_1 + h_2 \neq 0. \end{cases} \tag{6.4}$$

Remark. We recall that (6.3a) is the Jordanian subcase which produces the $GL_{h,h}(2)$ deformation of $GL(2)$. Thus, the three subcases of Hietarinta's R-matrix $R_{H2,3}$ are distiguished not only and not so much by the algebras they produce but intrinsically by their minimal polynomials.

To derive the corresponding quantum planes we shall apply the formalism of [41] (cf also [42]). The commutation relations between the coordinates z^i and differentials ζ^i ($i = 1, 2$) are given as follows:

$$z^i z^j \;=\; \mathcal{P}_{ijk\ell}\, z^k z^\ell \tag{6.5}$$

$$\zeta^i \zeta^j \;=\; -\mathcal{Q}_{ijk\ell}\, \zeta^k \zeta^\ell \tag{6.6}$$

$$z^i \zeta^j \;=\; \mathcal{Q}_{ijk\ell}\, \zeta^k z^\ell \tag{6.7}$$

where the operators $\mathcal{P}$, $\mathcal{Q}$ are functions of $\hat{R}$ and must satisfy

$$(\mathcal{P} - \mathbf{1})\,(\mathcal{Q} + \mathbf{1}) \;=\; 0. \tag{6.8}$$

In the well studied deformations of $GL(2)$ there are quadratic minimal polynomials and there are only two choices for the operators $\mathcal{P}$, $\mathcal{Q}$ (cf e.g.(6.3a). Here we have more choices. In particular, for the case (6.3b) we have four choices:

$$(\mathcal{P} - \mathbf{1},\ \mathcal{Q} + \mathbf{1}) = \begin{cases} (\hat{R} - \mathbf{1},\ \hat{R}^2 - \mathbf{1}) \\ (\hat{R} + \mathbf{1},\ (\hat{R} - \mathbf{1})^2) \\ (\hat{R}^2 - \mathbf{1},\ \hat{R} - \mathbf{1}) \\ ((\hat{R} - \mathbf{1})^2,\ \hat{R} + \mathbf{1}) \end{cases} \tag{6.9}$$

while in the case (6.3c) we have six choices:

$$(P - 1, \; Q + 1) = \begin{cases} \left(\hat{R} - 1, \; (\hat{R}^2 - 1)(\hat{R} - 1) \right) \\ \left(\hat{R} + 1, \; (\hat{R} - 1)^3 \right) \\ \left(\hat{R}^2 - 1, \; (\hat{R} - 1)^2 \right) \\ \left((\hat{R} - 1)^2, \; \hat{R}^2 - 1 \right) \\ \left((\hat{R}^2 - 1)(\hat{R} - 1), \; \hat{R} - 1 \right) \\ \left((\hat{R} - 1)^3, \; \hat{R} + 1 \right). \end{cases} \tag{6.10}$$

Our choice will be the last possibility of both (6.9) and (6.10), i.e. we shall use $P - 1 = (\hat{R} - 1)^a$ with $a = 2, 3$, respectively, and $Q = \hat{R}$ in all cases. With this choice and denoting $(x, y) = (z^1, z^2)$ we obtain from (6.5)

$$xy - yx = hy^2 \qquad h_1 = -h_2 = h \qquad P - 1 = (\hat{R} - 1)^2 \tag{6.11}$$

or

$$xy - yx = \tfrac{1}{2}(h_1 - h_2)y^2 \qquad h_1 \neq -h_2 \qquad P - 1 = (\hat{R} - 1)^3. \tag{6.12}$$

We note that the quantum planes corresponding to the bialgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are not essentially different. Furthermore the quantum plane (6.11) is the same as for the Jordanian subcase if we choose $P - 1 = \hat{R} - 1$.

Denoting $(\xi, \eta) = (\zeta^1, \zeta^2)$ we obtain from (6.6) with $Q = \hat{R}$

$$\xi^2 + \frac{h_1 - h_2}{2} \, \xi\eta \; = \; 0 \tag{6.13a}$$

$$\eta^2 \; = \; 0 \tag{6.13b}$$

$$\xi\eta \; = \; -\eta\xi. \tag{6.13c}$$

Of course, for $\hat{R} = P R_1$ (6.13a) simplifies to

$$\xi^2 + h \, \xi\eta = 0 \tag{6.14}$$

which is valid also for the Jordanian subcase.

Finally, for the coordinate-differential relations we obtain from (6.7) with $Q = \hat{R}$ again for all subcases

$$x\xi \; = \; \xi x + h_1\xi y + h_2\eta x + h_3\eta y \tag{6.15a}$$

$$x\eta \; = \; \eta x + h_1\eta y \tag{6.15b}$$

$$y\xi \; = \; \xi y + h_2\eta y \tag{6.15c}$$

$$y\eta \; = \; \eta y. \tag{6.15d}$$

Finally we derive the quantum plane relations for the case of the R_3 matrix. It is easy to see that (6.3b) holds also in this case, i.e. for

$$\hat{R}_3 \; \equiv \; P R_3 \; = \; \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6.16}$$

Using (6.5)–(6.7) with $P - 1 = (\hat{R}_3 - 1)^2$, $Q = \hat{R}_3$, we obtain, respectively,

$$xy \; = \; -yx \tag{6.17}$$

$$\xi^2 \; = \; 0 \tag{6.18a}$$

$$\eta^2 \; = \; 0 \tag{6.18b}$$

$$\xi\eta \; = \; \eta\xi \tag{6.18c}$$

$$x\xi = \xi x + \eta y \tag{6.19a}$$

$$x\eta = -\eta x \tag{6.19b}$$

$$y\xi = -\xi y \tag{6.19c}$$

$$y\eta = \eta y. \tag{6.19d}$$

Finally, we note that a check of consistency of this formalism is to implement Manin's approach to quantum planes [7]. Namely, one takes quantum matrix T (cf (2.1)) as the transformation matrix of the two-dimensional quantum planes. This means that if we define

$$z'^i = T_{ij} z^j \qquad \zeta'^i = T_{ij} \zeta^j \tag{6.20}$$

then $(x', y') = (z'^1, z'^2)$ and $(\xi', \eta') = (\zeta'^1, \zeta'^2)$ should satisfy the same relations as (x, y) and (ξ, η). The latter statement may be used to recover the algebraic relations of the bialgebras. Namely, suppose that relations (6.11), (6.13b), (6.13c), (6.14) and (6.15), or relations (6.12), (6.13) and (6.15), or relations (6.17)–(6.19), hold for both (x, y) and (ξ, η), (x', y') and (ξ', η'); then substitute the expressions for (x', y') and (ξ', η') into these relations, under the assumption that a, b, c, d commute with (x, y) and (ξ, η); then the coefficients of the independent bilinears that may be built from (x, y) and (ξ, η) will reproduce the algebraic relations of the bialgebras $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$.

7. Conclusions and outlook

In this paper we have found the bialgebras which are in duality with the three exotic matrix bialgebras (obtained in [36]) which are *not* deformations of the classical algebra of functions over the group $GL(2)$ or the supergroup $GL(1|1)$.

These bialgebras are rather degenerate and by their example we discover several hitherto unknown phenomena. To illustrate this we comment in more detail on the first two cases (considered in sections 3 and 4). The starting point is the 4×4 R-matrices R_1 and R_2 (cf (3.2), (4.1)). On the one hand R_1 is a special case of R_2 obtained for

$$h_1 = -h_2 = h. \tag{7.1}$$

On the other hand the algebraic relations obtained in [36] by applying the FRT formalism are different and those for $\mathcal{A}_1$ (3.7) cannot be obtained from those for $\mathcal{A}_2$ (4.4) by using (7.1). Another peculiarity is that the parameter h_3 does not appear either in (3.7), or in (4.4). However, the dependence on the parameters turns out to be redundant in both cases, after we find the bialgebras they are in duality with, namely, $\mathcal{U}_1$ and $\mathcal{U}_2$, respectively. This is so since the parameter dependence enters only through the (co-) ideals of the bialgebras, which (co-) ideals have zero pairings with $\mathcal{U}_1$ and $\mathcal{U}_2$, respectively. Thus, these (co-) ideals can be neglected (or we can pass to factor-algebras). As a result the reduced bialgebras coincide and the same holds for their bialgebras in duality. The two algebras differ, though only in a limited sense, if we also use the L–T duality of FRT [40], cf (3.29), the limited sense being that the L operators have the same parameter dependence as the matrices R_1 and R_2, and in the same way the L operators for $\mathcal{U}_1$ can be obtained from the L operators of $\mathcal{U}_2$ by (7.1).

Thus, the FRT formalism is more stringent since it preserves the initial parameter dependence. On the other hand, it reproduces only two of the basic generators of $\mathcal{U}_1$ and $\mathcal{U}_2$, namely, those that we denote by $\tilde{A}, B$, and gives no information on the other two basic generators $C, \tilde{D}$. This insufficiency of the FRT formalism is similar to that observed for the Jordanian deformations (cf e.g. [16]).

To conclude, the real difference between the two cases is exhibited by only the minimal polynomials of the permuted R-matrices. The significance of this will be revealed fully in the representation theory of the exotic bialgebras and their duals, which we intend to develop next.

Acknowledgment

References

[1] Demidov E E, Manin Yu I, Mukhin E E and Zhdanovich D V 1990 Nonstandard quantum deformations of $GL(n)$ and constant solutions of the Yang–Baxter equation *Prog. Theor. Phys. Suppl.* **102** 203

[2] Aghamohammadi A 1993 The two-parameter extension of h deformation of $GL(2)$, and the differential calculus on its quantum plane *Mod. Phys. Lett.* A **8** 2607

[3] Hinrichsen H and Rittenberg V 1992 A two parameter deformation of the $SU(1|1)$ superalgebra and the XY quantum chain in a magnetic field *Phys. Lett.* B **275** 350

[4] Dabrowski L and Wang L 1991 Two parameter quantum deformation of $GL(1|1)$ *Phys. Lett.* B **266** 51

[5] Burdik C and Tomasek R 1992 The two parameter deformation of the supergroup $GL(1|1)$, its differential calculus and its Lie algebra *Lett. Math. Phys.* **26** 97

[6] Frappat L, Hussin V and Rideau G 1998 Classification of the quantum deformations of the superalgebra $GL(1|1)$ *J. Phys. A: Math. Gen.* **31** 4049

[7] Manin Yu I 1987 *Ann. Inst. Fourier* **37** 191–205

[8] Corrigan E, Fairlie D B, Fletcher P and Sasaki R 1990 *J. Math. Phys.* **31** 776

[9] Schirrmacher A, Wess J and Zumino B 1991 The two parameter deformation of GL(2), its differential calculus, and Lie algebra *Z. Phys.* C **49** 317

[10] Ewen H, Ogievetsky O and Wess J 1991 Quantum matrices in two-dimensions *Lett. Math. Phys.* **22** 297

[11] Ogievetsky O and Wess J 1991 Relations between $GL(p, q)(2)$'s *Z. Phys.* C **50** 123

[12] Sudbery A 1991 Non-commuting coordinates and differential operators *Quantum Groups: Proc. ANL Workshop (Argonne, 1990)* ed T Curtright, D Fairlie and C Zachos (Singapore: World Scientific) p 33

[13] Zakrzewski S 1991 A Hopf star-algebra of polynomials on the quantum $SL(2, R)$ for a 'unitary' R-matrix *Lett. Math. Phys.* **22** 287

[14] Dobrev V K 1992 Duality for the matrix quantum group $GL_{p,q}(2, \mathbb{C})$ *J. Math. Phys.* **33** 3419

[15] Kupershmidt B A 1992 Classification of the quantum group structures on the group $GL(2)$ *J. Phys. A: Math. Gen.* **25** L1239

[16] Ohn C 1929 A $*$-product on $SL(2)$ and the corresponding nonstandard quantum $U(sl(2))$ *Lett. Math. Phys.* **25** 85–8

[17] Fairlie D and Zachos C 1991 Quantized planes and multiparameter deformations of Heisenberg and $GL(N)$ algebras *Quantum Field Theory, Statistical Mechanics, Quantum Groups and Topology: Proc. NATO Workshop (Miami, 1991)* ed T Curtright *et al* (Singapore: World Scientific) p 81

[18] Vladimirov A A 1993 A closed expression for the universal R matrix in a nonstandard quantum double *Mod. Phys. Lett.* A **8** 2573

[19] Ogievetsky O 1994 Hopf structures on the Borel subalgebra of $SL(2)$ *Proc. Winter School 'Geometry and Physics' (Zhidkov) (Suppl. Rendiconti Circolo Matematici di Palermo, Serie II vol 37)* p 185

[20] Karimipour V 1995 Bicovariant differential geometry of the quantum group $SL_h(2)$ *Lett. Math. Phys.* **35** 303

[21] Schmüdgen K and Schüler A 1996 Left-covariant differential calculi on $SL_q(2)$ and $SL_q(3)$ *Preprint* q-alg/9601020
Schmüdgen K and Schüler A 1996 *NTZ-preprint* 31/1995

[22] Ballesteros A, Celeghini E, Herranz F J, del Olmo M A and Santander M 1995 Universal R-matrices for non-standard $(1 + 1)$ quantum groups *J. Phys. A: Math. Gen.* **28** 3129

[23] Jagannathan R and Van der Jeugt J 1995 Finite dimensional representations of the quantum group $GL_{p,q}(2)$ using the exponential map from $U_{p,q}(gl(2))$ *J. Phys. A: Math. Gen.* **28** 2819

[24] Abdesselam B, Beckers J, Chakrabarti A and Debergh N 1996 On a deformation of $sl(2)$ with paragrassmannian variables *J. Phys. A: Math. Gen.* **29** 6729

[25] Dobrev V K 1996 Representations of the Jordanian quantum algebra $U_h(sl(2))$ *Proc. 10th Int. Conf. on Problems of Quantum Field Theory (Alushta, 1996)* JINR E2-96-369, ed D Shirkov, D Kazakov and A Vladimirov (Dubna) pp 104–10

[26] Chakrabarti R and Jagannathan R 1996 On the Hopf structure of $U_{p,q}(gl(1|1))$ and the universal $\mathcal{T}$-matrix of $Fun_{p,q}(GL(1|1))$ *Lett. Math. Phys.* **37** 191

[27] Aneva B L, Dobrev V K and Mihov S G 1997 Duality for the Jordanian matrix quantum group $GL_{g,h}(2)$ *J. Phys. A: Math. Gen.* **30** 6769

[28] Kondratowicz P and Podles P 1997 On representation theory of quantum $SL_q(2)$ groups at roots of unity *Banach Center Publication* vol 40, p 223

[29] Abdesselam B, Chakrabarti A and Chakrabarti R 1998 Towards a general construction of nonstandard R_h-matrices as contraction limits of R_q-matrices: the $\mathcal{U}_h(sl(N))$ algebra case *Mod. Phys. Lett.* A **13** 779

[30] Jacobs A D and Cornwell J F 1998 Classification of bicovariant differential calculi on the Jordanian quantum groups $GL_{g,h}(2)$ and $SL_h(2)$ and quantum Lie algebras *Preprint* math.QA/9802081

[31] Abdesselam B, Chakrabarti A, Chakrabarti R and Segar J 1999 Maps and twists relating $\mathcal{U}(sl(2))$ and the nonstandard $\mathcal{U}_h(sl(2))$: unified construction *Mod. Phys. Lett.* A **14** 765

[32] Chakrabarti R and Quesne C 1999 On Jordanian $\mathcal{U}_{q,h}(gl(2))$ algebra and its T matrices via a contraction method *Int. J. Mod. Phys.* A **14** 2511

[33] Ballesteros A, Herranz F J and Parashar P 1998 Multiparametric quantum $gl(2)$: Lie bialgebras, quantum R-matrices and non-relativistic limits *Preprint* math.QA/9806149
    Ballesteros A, Herranz F J and Parashar P 1999 *J. Phys. A: Math. Gen.* **32** 2369

[34] Aizawa N 1999 Symplecton for $U_h(sl(2))$ and representations of $SL_h(2)$ *J. Math. Phys.* **40** 5921
    Aizawa N 2000 Representation functions for Jordanian quantum group $SL_h(2)$ and Jacobi polynomials *J. Phys. A: Math. Gen.* **33** 3735

[35] Chakrabarti A and Chakrabarti R 2000 The Gervais–Neveu–Felder equation for the quasi-Hopf $\mathcal{U}_{h;y}(sl(2))$ algebra *J. Phys. A: Math. Gen.* **33** 1
    (Chakrabarti A and Chakrabarti R 2000 *Preprint* math.QA/0001015)

[36] Aneva B L, Arnaudon D, Chakrabarti A, Dobrev V K and Mihov S G 2001 On combined standard–nonstandard or hybrid (q,h)-deformations *J. Math. Phys.* **42** 1236
    (Aneva B L, Arnaudon D, Chakrabarti A, Dobrev V K and Mihov S G 2000 *Preprint* math.QA/0006206, INRNE-TH-00-02, LAPTH-800/00, S 080.0600, UNN-SCM-M-00-04)

[37] Chakrabarti A 2000 *RTT* relations, a modified braid equation and noncommutative planes *Preprint* math.QA/0009178 (*J. Math. Phys.* at press)

[38] Lyakhovsky V, Mirolubov A and del Olmo M 2000 Quantum Jordanian twist *Preprint* math.QA/0010198
    Lyakhovsky V, Mirolubov A and del Olmo M 2000 *Preprint* SPBU-IP-00-28

[39] Hietarinta J 1993 Solving the two-dimensional constant quantum Yang–Baxter equation *J. Math. Phys.* **34** 1725

[40] Faddeev L D, Reshetikhin N Yu and Takhtajan L A 1989 Quantization of Lie groups and Lie algebras *Alg. Anal.* **1** 178 (in Russian) (Engl. Transl. 1990 *Leningrad. Math. J.* **1** 193)
    see also 1998 *Algebraic Analysis* vol 1 (New York: Academic) p 129

[41] Wess J and Zumino B 1990 Covariant differential calculus on the quantum hyperplane *Nucl. Phys.* (Proc. Suppl.) **18** 302

[42] Schwenk J 1991 Differential calculus for the n-dimensional quantum plane *Quantum Groups: Proc. ANL Workshop (Argonne, 1990)* ed T Curtright, D Fairlie and C Zachos (Singapore: World Scientific) p 53

[43] Abe E 1980 Hopf algebras *Cambridge Tracts in Mathematics* N 74 (Cambridge: Cambridge University Press)

[44] Dobrev V K and Parashar P 1993 Duality for multiparametric quantum $GL(n)$ *J. Phys. A: Math. Gen.* **26** 6991–7002
    Dobrev V K and Parashar P 1999 *J. Phys. A: Math. Gen.* **32** 443 (addendum)

[45] Dobrev V K and Parashar P 1993 Duality for a Lorentz quantum group *Lett. Math. Phys.* **29** 259

[46] Podles and Woronowicz S L 1990 *Commun. Math. Phys.* **130** 381
    Carow-Watamura U, Schlieker M, Scholl M and Watamura S 1990 *Z. Phys.* C **48** 159

[47] Dobrev V K and Tahri E H 1998 Duality for multiparametric quantum deformation of the supergroup $GL(m/n)$ *Int. J. Mod. Phys.* A **13** 4339–66